

CSC 311: Introduction to Machine Learning

Lecture 7 - Probabilistic Models

Amanjit Singh Kainth

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Outline

- 1 Probabilistic Modeling of Data
- 2 Discriminative and Generative Classifiers
- 3 Naïve Bayes Models
- 4 Bayesian Parameter Estimation

- So far in the course we have adopted a modular perspective, in which the model, loss function, optimizer, and regularizer are specified separately.
- Today we begin putting together a **probabilistic interpretation** of our model and loss, and introduce the concept of **maximum likelihood estimation**.

Probabilistic Modeling of Data

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Example: A Biased Coin

You flip a coin $N = 100$ times and get outcomes $\{x_1, \dots, x_N\}$ where $x_i \in \{0, 1\}$ and $x_i = 1$ is interpreted as heads H .

Suppose you had $N_H = 55$ heads and $N_T = 45$ tails.

We want to create a model to predict the outcome of the next coin flip. That is, we want to answer this question:

What is the probability it will come up heads if we flip again?

Model

The coin may be biased. Let's assume that one coin flip outcome x is a **Bernoulli random variable** for a *currently unknown parameter* $\theta \in [0, 1]$.

$$p(x = 1|\theta) = \theta \quad \text{and} \quad p(x = 0|\theta) = 1 - \theta$$

or more succinctly $p(x|\theta) = \theta^x(1 - \theta)^{1-x}$

Assume that $\{x_1, \dots, x_N\}$ are **independent and identically distributed (i.i.d.)**. Thus, the joint probability of the outcome $\{x_1, \dots, x_N\}$ is

$$p(x_1, \dots, x_N|\theta) = \prod_{i=1}^N \theta^{x_i} (1 - \theta)^{1-x_i}$$

Loss Function

The **likelihood function** is the probability of observing the data as a function of the parameters θ :

$$L(\theta) = \prod_{i=1}^N \theta^{x_i} (1 - \theta)^{1-x_i}$$

We usually work with log-likelihoods (why?):

$$\ell(\theta) = \sum_{i=1}^N x_i \log \theta + (1 - x_i) \log(1 - \theta)$$

Maximum Likelihood Estimation

How can we choose θ ? Good values of θ should assign high probability to the observed data.

The **maximum likelihood criterion** says that we should pick the parameters that maximize the likelihood.

$$\hat{\theta}_{\text{ML}} = \arg \max_{\theta \in [0,1]} \ell(\theta)$$

We can find the optimal solution by setting derivatives to zero.

$$\frac{d\ell}{d\theta} = \frac{d}{d\theta} \left(\sum_{i=1}^N x_i \log \theta + (1 - x_i) \log(1 - \theta) \right) = \frac{N_H}{\theta} - \frac{N_T}{1 - \theta}$$

where $N_H = \sum_i x_i$ and $N_T = N - \sum_i x_i$.

Setting this to zero gives the maximum likelihood estimate:

$$\hat{\theta}_{\text{ML}} = \frac{N_H}{N_H + N_T}.$$

Maximum Likelihood Estimation

- define a model that assigns a probability (or has a probability density at) to a dataset
- maximize the likelihood (or minimize the neg. log-likelihood).

Discriminative and Generative Classifiers

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Spam Classification

For a large company that runs an email service, one of the important predictive problems is the automated detection of spam email.



Dear Karim,

I think we should postpone the board meeting to be held after Thanksgiving.

Regards,
Anna

Not spam



Dear Toby,

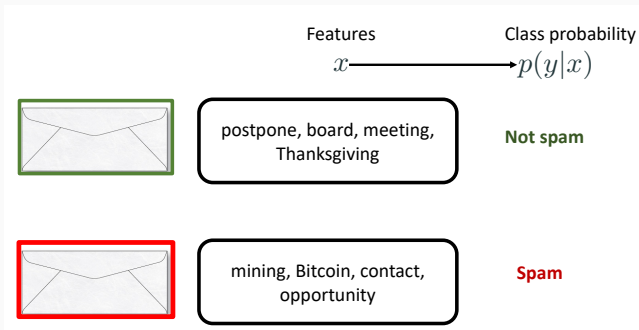
I have an incredible opportunity for mining 2 Bitcoin a day. Please Contact me at the earliest at +1 123 321 1555. You won't want to miss out on this opportunity.

Regards,
Ark

Spam

Discriminative Classifiers

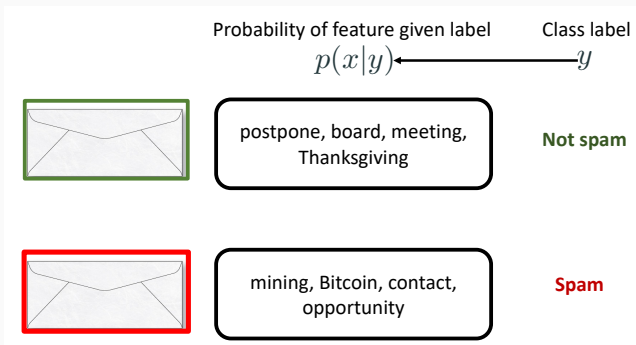
Discriminative classifiers try to learn mappings directly from the space of inputs \mathcal{X} to class labels $\{0, 1, 2, \dots, K\}$



Generative Classifiers

Generative classifiers try to build a model of “what data for a class looks like”, i.e. model $p(\mathbf{x}, y)$. If we know $p(y)$ we can easily compute $p(\mathbf{x}|y)$.

Classification via Bayes rule (thus also called Bayes classifiers)



Generative vs Discriminative

- **Discriminative approach:** estimate parameters of decision boundary/class separator directly from labeled examples.
 - ▶ Model $p(t|\mathbf{x})$ directly (logistic regression models)
 - ▶ Learn mappings from inputs to classes (linear/logistic regression, decision trees etc)
 - ▶ Tries to solve: How do I separate the classes?
- **Generative approach:** model the distribution of inputs characteristic of the class (Bayes classifier).
 - ▶ Model $p(\mathbf{x}|t)$
 - ▶ Apply Bayes Rule to derive $p(t|\mathbf{x})$.
 - ▶ Tries to solve: What does each class "look" like?
- Key difference: is there a distributional assumption over inputs?

Naïve Bayes Models

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Example: Spam Detection

- Classify email into spam ($c = 1$) or non-spam ($c = 0$).
- Binary features $\mathbf{x} = [x_1, \dots, x_D]$, $x_i \in \{0, 1\}$ saying whether each of D words appears in the e-mail.

Example email: “You are one of the very few who have been selected as a winner for the free \$1000 Gift Card.”

Feature vector for this email:

- ...
- “card”: 1
- ...
- “winners”: 1
- “winter”: 0
- ...
- “you”: 1

Bayesian Classifier

Given features $\mathbf{x} = [x_1, x_2, \dots, x_D]^T$

want to compute class probabilities using Bayes Rule:

$$\underbrace{p(c|\mathbf{x})}_{\text{Pr. class given feature}} = \frac{\overbrace{p(\mathbf{x}|c)}^{\text{Pr. feature given class}} p(c)}{p(\mathbf{x})}$$

In words,

$$\text{Posterior for class} = \frac{\text{Pr. of feature given class} \times \text{Prior for class}}{\text{Pr. of feature}}$$

To compute $p(c|\mathbf{x})$ we need: $p(\mathbf{x}|c)$ and $p(c)$.

Motivation for Compact Representation

- Two classes: $c \in \{0, 1\}$.
- Binary features $\mathbf{x} = [x_1, \dots, x_D], x_i \in \{0, 1\}$
- Define a joint distribution $p(c, x_1, \dots, x_D)$.
How many probabilities do we need to specify this joint dist.?
- Let's impose **structure** on the distribution so that the representation is **compact** and allows for efficient **learning** and **inference**

Naïve Bayes Independence Assumption

Naïve assumption:

the features x_i are **conditionally independent** given the class c .

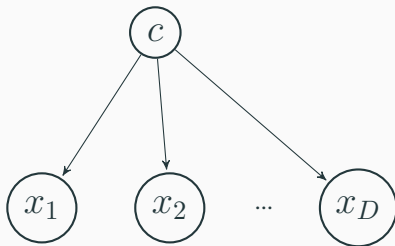
- Allows us to decompose the joint distribution:

$$p(c, x_1, \dots, x_D) = p(c) p(x_1|c) \cdots p(x_D|c).$$

Compact representation of the joint distribution

- Prior probability of class:
 $p(c = 1) = \pi$ (e.g. prob of spam)
- Conditional probability of feature given class:
 $p(x_j = 1|c) = \theta_{jc}$ (e.g. prob of word appearing in spam)

Bayesian Network for a Naive Bayes Model



We can form a graphical model.

- Which probabilities do we need to specify this dist.?
- How many probabilities do we need to specify this dist.?

Decomposing the Log-Likelihood

Decompose the log-likelihood into independent terms.

Optimize each term independently.

$$\begin{aligned}\ell(\boldsymbol{\theta}) &= \sum_{i=1}^N \log p(c^{(i)}, \mathbf{x}^{(i)}) = \sum_{i=1}^N \log \left\{ p(\mathbf{x}^{(i)} | c^{(i)}) p(c^{(i)}) \right\} \\ &= \sum_{i=1}^N \log \left\{ p(c^{(i)}) \prod_{j=1}^D p(x_j^{(i)} | c^{(i)}) \right\} \\ &= \sum_{i=1}^N \left[\log p(c^{(i)}) + \sum_{j=1}^D \log p(x_j^{(i)} | c^{(i)}) \right] \\ &= \underbrace{\sum_{i=1}^N \log p(c^{(i)})}_{\text{Log-likelihood of labels}} + \sum_{j=1}^D \underbrace{\sum_{i=1}^N \log p(x_j^{(i)} | c^{(i)})}_{\text{Log-likelihood for feature } x_j}\end{aligned}$$

Learning the Prior over Class

- To learn the prior, we maximize $\sum_{i=1}^N \log p(c^{(i)})$
- Define $\pi = p(c^{(i)} = 1)$
- Pr. i -th email: $p(c^{(i)}) = \pi^{c^{(i)}} (1 - \pi)^{1-c^{(i)}}$.
- Log-likelihood of the dataset:

$$\sum_{i=1}^N \log p(c^{(i)}) = \sum_{i=1}^N c^{(i)} \log \pi + \sum_{i=1}^N (1 - c^{(i)}) \log(1 - \pi)$$

- Maximum likelihood estimate of the prior π is the fraction of spams in dataset.

$$\hat{\pi} = \frac{\sum_i \mathbb{1}[c^{(i)} = 1]}{N} = \frac{\# \text{ spams in dataset}}{\text{total \# samples}}$$

Learning Pr. Feature Given Class

- To learn $p(x_j^{(i)} = 1 | c)$, we maximize $\sum_{i=1}^N \log p(x_j^{(i)} | c^{(i)})$
- Define $\theta_{jc} = p(x_j^{(i)} = 1 | c)$.
- Pr. of i -th email: $p(x_j^{(i)} | c) = \theta_{jc}^{x_j^{(i)}} (1 - \theta_{jc})^{1-x_j^{(i)}}$.
- Log-likelihood of the dataset:

$$\begin{aligned} \sum_{i=1}^N \log p(x_j^{(i)} | c^{(i)}) &= \sum_{i=1}^N c^{(i)} \left\{ x_j^{(i)} \log \theta_{j1} + (1 - x_j^{(i)}) \log(1 - \theta_{j1}) \right\} \\ &\quad + \sum_{i=1}^N (1 - c^{(i)}) \left\{ x_j^{(i)} \log \theta_{j0} + (1 - x_j^{(i)}) \log(1 - \theta_{j0}) \right\} \end{aligned}$$

- Maximum likelihood estimate of θ_{jc}
is the fraction of word j occurrences in each class in the dataset.

$$\hat{\theta}_{jc} = \frac{\sum_i \mathbb{1}[x_j^{(i)} = 1 \ \& \ c^{(i)} = c]}{\sum_i \mathbb{1}[c^{(i)} = c]} \quad \text{for } \underline{c=1} \quad \frac{\text{\#word } j \text{ appears in class } c}{\text{\# class } c \text{ in dataset}}$$

Predicting the Most Likely Class

- We predict the class by performing **inference** in the model.
- Apply **Bayes' Rule**:

$$p(c | \mathbf{x}) = \frac{p(c)p(\mathbf{x} | c)}{\sum_{c'} p(c')p(\mathbf{x} | c')} = \frac{p(c) \prod_{j=1}^D p(x_j | c)}{\sum_{c'} p(c') \prod_{j=1}^D p(x_j | c')}$$

- For input \mathbf{x} , predict c with the largest $p(c) \prod_{j=1}^D p(x_j | c)$
(the most likely class).

$$p(c | \mathbf{x}) \propto p(c) \prod_{j=1}^D p(x_j | c)$$

Naïve Bayes Properties

- An amazingly cheap learning algorithm!
- **Training time:** estimate parameters using maximum likelihood
 - ▶ Compute co-occurrence counts of each feature with the labels.
 - ▶ Requires only one pass through the data!
- **Test time:** apply Bayes' Rule
 - ▶ Cheap because of the model structure. (For more general models, Bayesian inference can be very expensive and/or complicated.)
- Analysis easily extends to prob. distributions other than Bernoulli.
- Less accurate in practice compared to discriminative models due to its “naïve” independence assumption.

Bayesian Parameter Estimation

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Data Sparsity

Maximum likelihood can overfit if there is too little data.

Example: what if you flip the coin twice and get H both times?

$$\theta_{\text{ML}} = \frac{N_H}{N_H + N_T} = \frac{2}{2 + 0} = 1$$

The model assigned probability 0 to T.
This problem is known as **data sparsity**.

Defining a Bayesian Model

We need to specify two distributions:

- The **prior distribution** $p(\boldsymbol{\theta})$
encodes our beliefs about the parameters
before we observe the data.
- The **likelihood** $p(\mathcal{D} | \boldsymbol{\theta})$
encodes the likelihood of observing the data
given the parameters.

The Posterior Distribution

- When we **update** our beliefs based on the observations, we compute the **posterior distribution** using Bayes' Rule:

$$p(\boldsymbol{\theta} | \mathcal{D}) = \frac{p(\boldsymbol{\theta})p(\mathcal{D} | \boldsymbol{\theta})}{\int p(\boldsymbol{\theta}')p(\mathcal{D} | \boldsymbol{\theta}') d\boldsymbol{\theta}'}$$

- Rarely ever compute the denominator explicitly.
- In general, computing the denominator is intractable.

Revisiting Coin Flip Example

We already know the likelihood:

$$L(\theta) = p(\mathcal{D}|\theta) = \theta^{N_H} (1 - \theta)^{N_T}$$

It remains to specify the prior $p(\theta)$.

- An **uninformative prior**, which assumes as little as possible. A reasonable choice is the uniform prior.
- But, experience tells us 0.5 is more likely than 0.99. One particularly useful prior is the **beta distribution**:

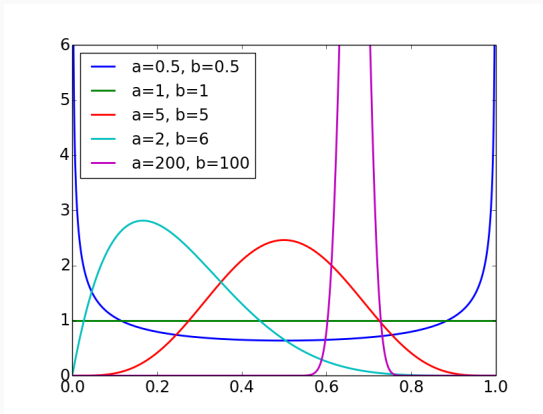
$$p(\theta; a, b) = \frac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)} \theta^{a-1} (1 - \theta)^{b-1}.$$

- We can ignore the normalization constant.

$$p(\theta; a, b) \propto \theta^{a-1} (1 - \theta)^{b-1}.$$

Beta Distribution Properties

- The expectation is $\mathbb{E}[\theta] = a/(a + b)$.
- The distribution gets more peaked when a and b are large.
- When $a = b = 1$, it becomes the uniform distribution.



Posterior for the Coin Flip Example

- Computing the posterior distribution:

$$\begin{aligned} p(\boldsymbol{\theta} | \mathcal{D}) &\propto p(\boldsymbol{\theta})p(\mathcal{D} | \boldsymbol{\theta}) \\ &\propto \left[\theta^{a-1}(1-\theta)^{b-1} \right] \left[\theta^{N_H}(1-\theta)^{N_T} \right] \\ &= \theta^{a-1+N_H}(1-\theta)^{b-1+N_T}. \end{aligned}$$

A beta distribution with parameters $N_H + a$ and $N_T + b$.

- The posterior expectation of θ is:

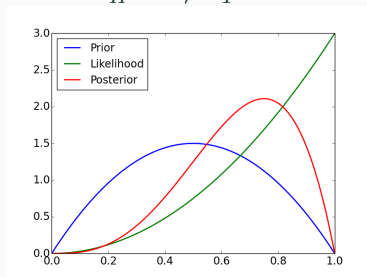
$$\mathbb{E}[\theta | \mathcal{D}] = \frac{N_H + a}{N_H + N_T + a + b}$$

- Think of a and b as **pseudo-counts**.
 $\text{beta}(a, b) = \text{beta}(1, 1) + a - 1$ heads + $b - 1$ tails.
- The prior and likelihood have the same functional form (conjugate priors).

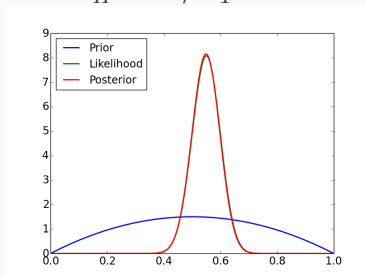
Bayesian Inference for the Coin Flip Example

When you have enough observations, the **data overwhelm the prior**.

Small data setting
 $N_H = 2, N_T = 0$

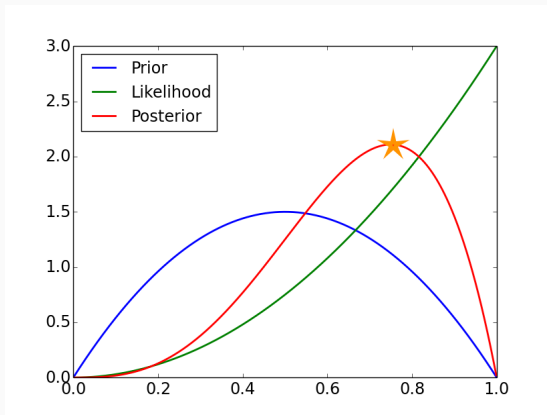


Large data setting
 $N_H = 55, N_T = 45$



Maximum A-Posteriori (MAP) Estimation

Finds the most likely parameters under the posterior (i.e. the mode).



Maximum A-Posteriori Estimation

Converts the Bayesian parameter estimation problem into a maximization problem

$$\begin{aligned}\hat{\boldsymbol{\theta}}_{\text{MAP}} &= \arg \max_{\boldsymbol{\theta}} p(\boldsymbol{\theta} | \mathcal{D}) \\ &= \arg \max_{\boldsymbol{\theta}} p(\boldsymbol{\theta}) p(\mathcal{D} | \boldsymbol{\theta}) \\ &= \arg \max_{\boldsymbol{\theta}} \log p(\boldsymbol{\theta}) + \log p(\mathcal{D} | \boldsymbol{\theta})\end{aligned}$$

Maximum A-Posteriori Estimation

Joint probability of parameters and data:

$$\begin{aligned}\log p(\theta, \mathcal{D}) &= \log p(\theta) + \log p(\mathcal{D} | \theta) \\ &= \text{Const} + (N_H + a - 1) \log \theta + (N_T + b - 1) \log(1 - \theta)\end{aligned}$$

Maximize by finding a critical point

$$\frac{d}{d\theta} \log p(\theta, \mathcal{D}) = \frac{N_H + a - 1}{\theta} - \frac{N_T + b - 1}{1 - \theta} = 0$$

Solving for θ ,

$$\hat{\theta}_{\text{MAP}} = \frac{N_H + a - 1}{N_H + N_T + a + b - 2}$$

Estimate Comparison for Coin Flip Example

	Formula	$N_H = 2, N_T = 0$	$N_H = 55, N_T = 45$
$\hat{\theta}_{\text{ML}}$	$\frac{N_H}{N_H + N_T}$	1	$\frac{55}{100} = 0.55$
$\mathbb{E}[\theta \mathcal{D}]$	$\frac{N_H + a}{N_H + N_T + a + b}$	$\frac{4}{6} \approx 0.67$	$\frac{57}{104} \approx 0.548$
$\hat{\theta}_{\text{MAP}}$	$\frac{N_H + a - 1}{N_H + N_T + a + b - 2}$	$\frac{3}{4} = 0.75$	$\frac{56}{102} \approx 0.549$

$\hat{\theta}_{\text{MAP}}$ assigns nonzero probabilities as long as $a, b > 1$.